Pareto-minimality in the jungle

(Revised Version)

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Abstract

We consider the simplest version of a jungle economy à la Piccione-Rubinstein where there are as many agents as goods, agents consume at most one indivisible good, and there is a transitive strong power relation. We first study how wild jungle equilibria are, *i.e.*, whether they are Pareto-minimal (an allocation is Pareto-minimal if it is impossible to decrease the welfare of one agent without increasing the welfare of another one). We show that jungle equilibria are not necessarily Pareto-minimal. We then study and characterize the set of Pareto-minimal jungle equilibria. Second, we tackle the case where people can be equally powerful, in contrast to the assumption that the power relation is asymetric. Assuming specifically that there is a transitive weak power relation we show that jungle equilibria exist, but they are not always unique, nor Pareto-optimal. We also provide conditions under which these equilibria are Pareto-minimal.

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1 Introduction

Exchanges are not always voluntary, nor mutually beneficial. While economists are still fascinated by the benefits provided by market forces and socially cooperative behavior (see, *e.g.*, Sugden, 2012, 2018), there is a large and growing literature on the conflictual allocation of resources.

In this literature, recently reviewed in Kimbrough *et al.* (2017), conflict "is a situation in which agents choose inputs that are (i) costly both to themselves and relatively to some socially efficient optimum (ii) in pursuit of private payoffs framed as wins and losses" (*ibid*, table 1, p. 4). In addition, the conflict issue is often conceived as a random variable which depends on the (Nash) equilibrium players' strategies.[1] For instance, in rent-seeking contest models, contestants exert costly efforts and their probability of winning a prize is given by a contest function whose arguments are precisely the contestants' efforts.[2]

Yet, allocation by force is not necessarily random. Power asymmetry and domination, argues Vahabi (2016 a, chapter 3), are neglected in modern contest theory. We have to acknowledge that sometimes the more powerful just takes from the less powerful. As Feldman and Serrano put it (2006, p. 79), "... You give me your wallet and your watch, or I'll kill you! This is the 'law of the jungle'." Jungle exchange is pervasive during wars or in predatory states. It is less so, at least in its extreme form, in modern democracies.

Allocation by force is notably taken up in Piccione and Rubinstein (2007) who focus on jungle exchange, in which economic agents use power to seize control of assets held by others. Piccione and Rubinstein introduce what they call a *jungle*. A jungle is a set of individuals, each having preferences over bundles of desirable goods, and a transitive strong power relation (it is irreflexive, asymmetric, total and transitive). In the jungle the exercise of power is costless: seizing the goods of others is free for the more powerful. It is also certain: there is no way the less powerful can escape coercion. A *jungle equilibrium* is then defined as a feasible allocation of goods such that no agent wants to take goods held by a weaker agent.

[1] This is not always the case. In Bush (1972) and Bush and Meyer (1974), two of the earliest papers on the economics of anarchy, the result of predation is certain.

[2] On contest success functions see, *e.g.*, Corchón and Serena (2018), or Pietri (2017).

Piccione and Rubinstein obtain two striking results. First, a jungle equilibrium is Pareto-optimal (this is the First Fundamental Theorem of Jungle Economics). Second, under some conditions, to each Pareto-optimal allocation corresponds a power relation for which the said allocation is a jungle equilibrium (they call this result the Second Fundamental Theorem of Jungle Economics). Interestingly, when agents have the same preferences and when certain conditions are met, there are prices supporting a jungle equilibrium as a competitive equilibrium and for these prices a stronger agent is also a richer one. Strength correlates with wealth.[3]

This paper furthers the analysis of jungle exchange, focusing on the simplest version of the jungle model. In this simplest version, there are n individuals and n indivisible objects and no agent can consume more than one object.[4] We take advantage of the simple setting to address two issues.

Firstly, we address the following question: how wild are jungle equilibria? One way to answer this question is to consider whether they are Pareto-minimal. An allocation is Pareto-minimal if it is impossible to decrease the welfare of an agent without *increasing* the welfare of another one (Luptáčik, 2009). A Pareto-minimal allocation has the dismal property that every possibility of simultaneously decreasing people's welfare has been exhausted. It turns out that jungle equilibria are not necessarily Pareto-minimal. We then study and characterize the set of Pareto-minimal jungle equilibria.

Secondly, we revisit the assumption that the power relation is asymetric. In particular, the property that there exists a strongest agent is not obvious. Consider for instance the following quotation of Hobbes who argues that men actually have equal power in the state of nature (Hobbes, 1981, [1651] p. 183).

For as to the strength of body, the weakest has strength enough to kill the

[3] Pursuing the line of research opened by Piccione and Rubinstein, Houba *et al.* (2017) study general conditions under which jungle equilibria coincide with lexicographic welfare maximizing allocations (in which all of the economy's resources are initially common goods and stronger agents take from the pile of common goods before weaker agents can).

[4] **This setting is identical to the model used by Feldman and Serrano (2006), chapter 4 (or Rubinstein, 2012), which is a special case of the general framework analyzed by Piccione and Rubinstein (2007). To avoid confusion, we shall hereafter mainly refer to the first model instead of the general jungle economy.**

> strongest, either by secret machination, or by the confederacy with others, that are in the same danger with himselfe.

While we shall neglect the "confederacy with others", we shall retain that physical strength can be counterbalanced by "secret", "machination", and wit. In this connection, it is no longer obvious that there always exists a strongest agent (and a second-strongest agent, and so on). Escape, for instance, is a way to avoid the control of a predatory state (for a broad view of the economics of escape see Vahabi, 2016 a). This is especially true for agents whose only wealth is their human capital (Vahabi, 2016 b). Thus, a predatory ruler may not be more powerful than his prey since allocation by force is impossible and, of course, the prey is not more powerful than his predator. In this sense, both the predatory ruler and his prey are equally powerful.

To account for these remarks we shall assume that there is a transitive weak power relation so that in contrast to the original jungle model people can be equally powerful. Under this assumption we show that jungle equilibria exist, need not be unique, and that not all of them are Pareto-optimal. We also characterize the set of Pareto-minimal jungle equilibria.

The paper unfolds as follows. In section 2 we briefly lay out a simple model of a jungle economy. Section 3 addresses the Pareto-minimality property of jungle equilibria when there is a transitive strong power relation. Section 4 considers jungle equilibria under the alternative assumption that there is a transitive weak power relation. Section 5 briefly concludes the paper.

2 Model

Following Feldman and Serrano (2006), assume that there is a bilateral *power relation*, denoted by $\succ$, among n individuals belonging to the set $I = \{1, 2, \ldots, n\}$ $(n \geq 2)$. They interpret the power relation as follows: if person i meets person j in the jungle, and $i \succ j$, then "person i can take whatever he wishes from j, whether j agrees on not." Feldman and Serrano assume that $\succ$ is a transitive strong relation (that is, it is total, asymmetric, and transitive).[5] Therefore, there exists a strongest agent, a second-strongest agent and so on.

The power relation does not necessarily refer to actual physical strength. Piccione and Rubinstein, *ibid*, make it clear that charm and attraction, as well as the power of conviction, are alternative means to exercise power and to control assets held by others.

After presenting the power relation, we consider in turn the endowments and the preferences of agents. We assume that each individual is endowed with one indivisible good and that each individual can consume one and only one good (no consumption at all is also possible).[6] The fact that an agent can consume at most one good embodies the idea of satiation, a property which, of course, ensures that should there be a Leviathan, he will not seize all the goods in the economy.[7] In addition, we suppose that the number of agents is exactly equal to the number of goods. We let $\mathcal{G}$ denote the set of goods, $\mathcal{G} = \{g_1, g_2, \ldots, g_n\}$.

Further, each individual is endowed with a real-valued utility function u_i defined on the set $\mathcal{G} \cup \{\emptyset\}$. Moreover, we assume that the preferences are strict (the utility function is injective).[8] We also assume that it is better to consume the good ranked the lowest rather than not consuming at all.

We shall occasionally assume that individual preferences are symmetric in the sense that the ordering of goods is the same for all agents.[9] We note in passing that according to Hobbes (Hobbes, 1981, [1651], p. 184) symmetry of preferences is one reason for the existence of conflict

> And therefore, if any two men desire the same thing, which nevertheless they cannot both enjoy, they become ennemies, and in the way to their end (which is principally their own conservation, and sometimes their delectation only), endeavor to destroy, or subdue, one another.

[5] These assumptions mean that it is always possible to compare the powers of two agents, that no two agents are equally powerful, and that if an agent i is more powerful than an agent j, and that agent j is more powerful than agent k, then agent i is also more powerful than agent k.

[6] The indivisible good commonly referred to is a house. This is reminiscent of the Shapley and Scarf (1974) paper.

[7] This is certainly a disputable assumption (but see section 2 of Piccione and Rubinstein 2007). It is, however, instrumental in simplifying the analysis.

[8] A function $f : \mathcal{C} \to \mathbb{R}$ is injective if it never maps two different elements of $\mathcal{C}$ to the same element of $\mathbb{R}$.

[9] This assumption is made in section 4.3 of Piccione and Rubinstein (2007).

An *allocation* $\boldsymbol{x}$ associates with each good one, and only one, individual. We let $x(i)$ be the good allocated to individual i.

Definition 1 (Feldman and Serrano, 2006)**.** *An allocation* $\boldsymbol{x}$ *is a jungle equilibrium such that, for every* i *and* j*, if agent* i *is more powerful than an agent* j*, i.e.,* $i \succ j$*, then* $u_i\big(x(i)\big) \geq u_i\big(x(j)\big)$.[10]

That is, in a jungle equilibrium if an agent is more powerful than another one, the more powerful agent holds a good that delivers at least as much utility as does the good held by the less powerful one. **Notice that since the utility functions are assumed to be injective there exists a unique jungle equilibrium (this is so because no agent is indifferent between two goods).** In the next section, we concentrate on a property of jungle equilibria, namely Pareto-minimality.

3 Pareto-Minimality in the Jungle

We first introduce the notion of Pareto-worsening.

Definition 2. *We say that an allocation* $\boldsymbol{y}$ *is a Pareto-worsening of an allocation* $\boldsymbol{x}$ *if for all agents* i *we have* $u_i\big(y(i)\big) \leq u_i\big(x(i)\big)$ *and there is an agent* k *such that* $u_k\big(y(k)\big) < u_k\big(x(k)\big)$*.*

The notion of Pareto-minimal allocation is defined as follows[11]

Definition 3. *An allocation is Pareto-minimal if it does not admit a Pareto-worsening.*

Hence, in a Pareto minimal allocation all the possibilities of simultaneously worsening the well-being of all the agents have been exhausted. Pareto-minimal allocation, therefore, can be associated with particularly dismal situations.

When agents' preferences are all the same, any allocation is Pareto-minimal. That is because starting with an arbitrary allocation, any alternative allocation involves at least one agent

[10] Notice that when the utility functions are injective the condition $u_i\big(x(i)\big) \geq u_i\big(x(j)\big)$ actually implies $u_i\big(x(i)\big) > u_i\big(x(j)\big)$.

[11] On the notion of Pareto-minimal point, see, *e.g.*, Luptáčik (2009) (p. 226).

that is better off. As an immediate corollary, when agents have the same preferences any jungle equilibrium is Pareto-minimal.

A jungle equilibrium, however, is not always Pareto-minimal. Consider the following example drawn from Feldman and Serrano (2006).

Example 1 Assume that there are four agents (1, 2, 3, 4) who rank four goods as follows:[12]

1	2	3	4
g_2	g_4	g_1	g_2
g_1	g_3	g_3	g_4
g_3	g_2	g_2	g_1
g_4	g_1	g_4	g_3

One can check that the allocation (g_2, g_4, g_1, g_3) is a jungle equilibrium (this is the unique one). But if we allocate good g_1 to player 2, and good g_4 to player 3, leaving the other agents with their equilibrium holdings, we decrease the payoffs of these two players. Thus the jungle equilibrium is not Pareto-minimal. This is an instance where the jungle is not so wild.

The following Proposition gives a characterization of Pareto-minimal jungle equilibria which shows that symmetric preferences are not necessary for a jungle equilibrium to be Pareto-minimal.

Proposition 1. *A jungle equilibrium is Pareto-minimal if, and only if, the following conditions on preferences hold*

$$u_1\big(x(1)\big) > \max\left\{u_1\big(x(k)\big), k \neq 1\right\},$$
$$\min\{u_i\big(x(k)\big), k = 1, \ldots, i-1\} > u_i\big(x(i)\big) > \max\{u_i\big(x(k)\big), k = i+1, \ldots, n\},\ \forall i \neq 1, n,$$
$$\min\{u_n\big(x(k)\big), k = 1, \ldots, n-1\} > u_n\big(x(n)\big).$$

Proof. Let us first consider the "only if" part. Let $\boldsymbol{x}$ be a jungle equilibrium and assume that $1 \succ 2 \succ, \ldots, \succ n$. Since the power relation is asymmetric, by definition of jungle equilibrium

[12] In this table column i describes the ranking of agent i.

we must have

$$u_1\big(x(1)\big) > \max\{u_1\big(x(k)\big), k \neq 1\}, \tag{1}$$

and for any agent $i > n$,

$$u_i\big(x(i)\big) > \max\{u_i\big(x(k)\big), k = i+1, \ldots, n\}. \tag{2}$$

Now since $\boldsymbol{x}$ is Pareto-minimal, we must also have: $u_i\big(x(1)\big) > u_i\big(x(i)\big)$, for $i \neq 1$. If this property does not hold for an agent i, then by exchanging the goods allocated to this agent and agent 1, leaving the remaining part of the equilibrium allocation unchanged, we would make the two agents worse off. This would contradict the assumption that the allocation is Pareto-minimal. It must also be that $u_i\big(x(k)\big) > u_i\big(x(i)\big)$, for all $k = 1, \ldots, i-1$, otherwise, by the same argument as above, we would contradict the assumption that $\boldsymbol{x}$ is Pareto-minimal. This establishes the "only if part". The "if part" is immediate. □

The above Proposition shows that in a Pareto-minimal equilibrium, any agent ranks goods in the same way as all the agents that are stronger than him. But the goods ranking is not necessarily the same from one agent to another. To see this consider the next example of a three-agent jungle where agent 3's ranking of goods differs from those of agents 1 and 2

Example 2 **One can check that (g_1, g_2, g_3) is a Pareto-minimal jungle equilbrium.**

1	2	3
g_1	g_1	g_2
g_2	g_2	g_1
g_3	g_3	g_3

Agent 3 would be better off with the goods held by agents 1 and 2 who are more powerful than him (but it does not follow that agent 3 must have the same preferences as agents 1 and 2).

While we have shown that there may be Pareto-minimal jungle equilibria, however, the definition of Pareto-minimality given at the beginning of this section is disputable. In this definition, we ask whether it is possible to worsen the payoff of some agents. But doing this is, of course, meaningless if it is impossible to force an agent to hold a good which causes a decrease in his welfare. For instance, in Example 1 we have seen that the jungle equilibrium is not Pareto-minimal since there is an exchange of goods which can diminish the utility of the two agents. But in this example agent 2 can oppose this exchange as he is more powerful than agent 3. This remark motivates the introduction of an alternative notion of Pareto-minimality. To do this, we begin with the following notion of a $\succ$-*worsening*.

Definition 4. *We say that an allocation* $\boldsymbol{y}$ *is a* $\succ$*-worsening of an allocation* $\boldsymbol{x}$ *if for all agents* i *we have* $u_i\big(y(i)\big) \leq u_i\big(x(i)\big)$ *and there are at least two agents* j *and* k*,* $j \succ k$*, such that* $y(k) = x(j)$ *and* $u_k\big(y(k)\big) < u_k\big(x(k)\big)$*.*

In other words, whenever it is possible to decrease agent k's welfare, there must be a more powerful agent who forces k to hold a less interesting good. We can now adapt our definition of Pareto-minimality by introducing the notion of $\succ$-Pareto-minimality.

Definition 5. *An allocation* $\boldsymbol{x}$ *is* $\succ$*-Pareto-minimal if it does not admit a worsening.*

When then have

Proposition 2. *Any jungle equilibrium is* $\succ$*-Pareto-minimal.*

Proof. Suppose that a jungle equilibrium $\boldsymbol{x}$ is not $\succ$-Pareto-minimal. Then there is an allocation $\boldsymbol{y}$ which satisfies: $u_i\big(y(i)\big) \leq u_i\big(x(i)\big)$ for all i, and there is an agent k for whom $u_k\big(y(k)\big) < u_j\big(x(k)\big)$ where $y(k) = x(j)$ for some agent j who is such that $j \succ k$. Of necessity, $k \neq 1$, because agent 1 is the most powerful and no one can force him to change his allocation. This implies that $u_1\big(y(1)\big) = u_1\big(x(1)\big)$. Since the utility functions are all injective, we have $x(1) = y(1)$. Suppose then, that $k = 2$. This implies that $j = 1$. But since $x(1) = y(1)$ this is impossible. Therefore $y(2) = x(2)$. Continuing this reasoning shows that $\boldsymbol{y} = \boldsymbol{x}$, which is a contradiction. □

4 Jungle equilibrium when there is a transitive weak power relation

In this section we revisit the assumption that the power relation is asymetric. We first consider the alternative assumption that there is a transitive weak power relation. We then pay attention to the existence of jungle equilibria. Finally we study whether these equilibria satisfy the Pareto-minimality property.

4.1 Transitive weak power relation

Up to now, we have assumed that the power relation is total, asymmetric, and transitive. In what follows we shall consider the case where the power relation is no longer asymmetric (some agents may have equal powers).

More specifically, we shall introduce a new binary power relation. We shall say that $i \gtrsim j$ if "agent i is at least as powerful as agent j". Then, we say that agent i is stronger (or more powerful) than j, or $i \succ j$, if $i \gtrsim j$ and $j \not\gtrsim i$.[13] Only in this case will agent i be able to take control of agent j's assets. Further, we will say that agents i and j are equally powerful whenever one has both $i \gtrsim j$ and $j \gtrsim i$. **In this case, we use the notation $i \sim j$. In any case, the relation will always be total**.

Of course, the transitivity property of $\gtrsim$ would not be satisfied where the power relation $\succ$ induced from $\gtrsim$ admits cycles. For instance, the power relation $\gtrsim$ in a 3-agent jungle has a cycle when: $1 \succ 2$, $2 \succ 3$ and $3 \succ 1$. In that case, $\gtrsim$ does not satisfy the transitivity property. As we shall see, however, transitivity of the power relation is preserved when the power relation $\gtrsim$ has only *weak cycles.*

Definition 6. *We say that the power relation has a weak cycle if there is a list of agents $i_1, \ldots, i_K$ such that: $i_1 \succ i_2 \succ ... \succ i_K$, and $i_K \sim i_1$.*

That is, a weak cycle is a list of agents where the first is more powerful than the second, the second more powerful than the third, and the first and last agents are equally powerful.

[13] The meaning of $j \not\gtrsim i$ is that it is false that j is at least as powerful as i.

To illustrate the idea of a weak-cycle, consider a country where there is a ruler, and a religious sect. The sect includes the sect leader, who is an old man, and a young man. Because the sect leader is old and weak and cannot move, the ruler is more powerful than him. The sect leader is also more powerful than the young man. On the other hand, as the young man can escape and flees the jungle, the ruler is not more powerful than him. In a sense these two people have equivalent power.

The connection between weak cycle and transitivity is made in the next result.

Proposition 3. *Assume that all the cycles of the power relation $\succsim$ are weak. Then $\succsim$ is a transitive weak power relation. That is, up to a renumbering of agents we have:* $1 \succsim 2 \succsim \cdots \succsim n-1 \succsim n$.

Proof. Let us first prove that there is an agent k who is at least as powerful as any other agent, namely, $k \succsim i$, for all i. Suppose this is false. Then, for any agent i, there is an agent j who is such that $j \succ i$. Now, let us pick up an agent and call him 1. We know that there is another agent, whom we call 2, such that $2 \succ 1$. Furthermore, there is also another agent, called 3, such that $3 \succ 2$ (necessarily, agent 3 is different from agent 1). Moreover, we cannot have $1 \succ 3$, because all the cycles are weak. Continuing this reasoning with the remaining agents, we get a list of agents such that: $n \succ n-1 \succ \cdots \succ i \succ i-1 \succ \ldots 3 \succ 2 \succ 1$. Then consider the $n-th$ agent. Since this agent cannot be at least as powerful as any agent, there is an agent i such that $i \succ n$. But this is impossible since all the cycles are weak. So there is an agent, say, agent 1, who is at least as powerful as any other agent. We can show similarly that there is an agent who is at least as powerful as any other agent, except perhaps agent 1. We call this agent 2. Repeatedly applying this argument to the other agents yields the result. □

The next example illustrates the above Proposition.

Example 3 Consider a 4-agent jungle where $\succsim$ satisfies: $1 \succ 2 \succ 3 \sim 1$ and $2 \succ 3 \succ 4 \sim 2$. The power relation admits two weak cycles. Let us show that there is a transitive weak relation. Since $\succsim$ is total we have either $1 \succsim 4$ or

$4 \succ 1$. **In the first case, we have** $1 \gtrsim i$, $i = 2, 3, 4$, $2 \gtrsim i$, $i = 3, 4$, $3 \gtrsim 4$. **Thus we have a transitive weak relation. If, on the other hand,** $4 \succ 1$, **then we have:** $1 \succ 2 \succ 3 \succ 4 \succ 1$**, and thus there is a cycle which is not weak, which is impossible by assumption.**[14]

From now on we shall assume that there is a transitive weak power relation, namely: $1 \gtrsim 2 \gtrsim \cdots \gtrsim n-1 \gtrsim n$. From Proposition 3, this is equivalent to assuming that all the cycles of the power relation are weak.

4.2 Existence of jungle equilibrium

We shall retain Feldman and Serrano (2006)'s definition of a *jungle equilibrium*. That is, in a jungle equilibrium, if one agent is more powerful than another one, the more powerful agent holds a good that delivers at least as much utility as the good held by the less powerful one. Notice that this property is not necessarily satisfied for equally powerful agents.[15]

Example 4 There are three agents and three goods. All the agents strictly prefer good 1 to good 2 and good 2 to good 3. Moreover, the power relation is as follows: $1 \succ 2$, $2 \succ 3$, $3 \sim 1$. One can check that the allocation (g_1, g_2, g_3) is a jungle equilibrium. This is true since we do not impose the condition $u_i\big(x(i)\big) \geq u_i\big(x(j)\big)$ whenever i and j are equally powerful.

Proposition 4. *When the power relation is transitive weak, there exists a jungle equilibrium.*

Proof. The proof is adapted from Feldman and Serrano (2006), who dealt with the case where there is a transitive strong power relation. Consider the agent (say 1) who is at least as powerful as any other agent. Give him his best choice (say $x(1)$). Then for any agent i such that $1 \succ i$, we have $u_1\big(x(1)\big) > u_1\big(x(i)\big)$. Next, consider the agent who is not agent 1, and who is at least as powerful as any other agent (except agent 1). Give this agent his best choice in the set of remaining available goods (that is, all the goods but $x(1)$). Calling this good $x(2)$ we observe that for any agent i such that $2 \succ i$, $i \neq 1$, $u_2\big(x(2)\big) > u_2\big(x(i)\big)$. We

[14] I thank a referee for suggesting studying this example.

[15] In particular, when all agents are equally powerful any allocation is a jungle equilibrium. This is, however, an extreme case, which can hardly correspond to what is generally referred to as a jungle.

can apply the same kind of argument by considering all the other agents in turn. Hence the allocation $\boldsymbol{x}$ is a jungle equilibrium. $\square$

The next Proposition relates symmetric preferences and transitive weak power relations.

Proposition 5. *Assume that preferences are symmetric and that there exists a jungle equilibrium. Then $\gtrsim$ is a transitive weak power relation.*

Proof. Consider the agent who holds good 1. Then this agent must be at least as powerful as any other agent. The same reasoning applies successively to all the other agents and the result follows. $\square$

When there is a transitive strong power relation and the utility functions are injective, there is a unique jungle equilibrium. By contrast, as the next example shows, when there is transitive weak power relation there may be several jungle equilibria.

Example 5 Consider the simplest setting one can think of. There are two agents who are equally powerful ($1 \sim 2$), and their tastes are as follows: $u_1(g_1) > u_1(g_2)$ and $u_2(g_2) > u_2(g_1)$. One can see that both allocations (g_1, g_2) and (g_2, g_1) are jungle equilibria.

Remark. Piccione and Rubinstein (2007) (subsection 3.3, p. 890) show that any jungle equilibrium generates a competitive equilibrium with no exchange. Furthermore, they show that in general there is no relationship between power and wealth: "In particular, if the strongest agent ranks the highest a house that all other agents rank the lowest, there exists a competitive equilibrium price vector in which the strongest agent is the poorest agent." When agents have the same preferences, however, Piccione and Rubinstein (*ibid*) show that "the relationship between power and wealth is unambiguous: the value of an agent i's jungle equilibrium bundle increases with his strength." Yet, this property is no longer true when there is a transitive weak power relation. Consider again example 4. In this example, associating any triple (p_1, p_2, p_3) of positive prices such as $p_3 < p_2 < p_1$ with the allocation (g_1, g_2, g_3) yields a competitive equilibrium. The poorest agent, agent 3, is nevertheless as powerful as the wealthiest agent, namely, agent 1. Power no longer correlates with wealth.

Inexistence of a jungle equilibrium

The following example illustrates what happens when there is no transitive weak power relation because $\gtrsim$ has a cycle which is not weak.

Example 6 There are three agents and three goods. All the agents strictly prefer good 1 to good 2 and good 2 to good 3. Moreover, the power relation is as follows: $1 \succ 2$, $2 \succ 3$, $3 \succ 1$. Then one can check that there is no jungle equilibrium. For instance, the following allocation (g_1, g_2, g_3) where good 1 is given to agent 1, good 2 to agent 2 and good 3 to agent 3 is not a jungle equilibrium. Indeed, since $3 \succ 1$, we should have $u_3(g_3) > u_3(g_1)$ but this is false by assumption.

Using this kind of argument, we can easily establish the following general result.

Proposition 6. *Assume that the power relation $\succ$ (induced from $\gtrsim$) is such that there is a list of agents $i_1, \ldots, i_K$ for whom: $i_1 \succ i_2 \succ ... \succ i_K$, and $i_K \succ i_1$. Then if agents' preferences are symmetric, there is no jungle equilibrium.*

This Proposition illustrates the effect of the instability of the power relation. This is an interesting result since, as we have already observed, the existence of a transitive strong power relation is disputable. Indeed, recall that the power relation does not necessarily refer to actual physical strength. For instance, agent 1 may have more physical strength than agent 2; agent 2 may have more charm than agent 3, and agent 3 may have a stronger power of conviction thant agent 1.

Pareto-optimality

Where there is a transitive strong power relation, Feldman and Serrano (2006, p. 89) shows that jungle equilibria are Pareto-optimal (this is the so-called First Fundamental Theorem of jungle economics). Example 5, however, shows that when the power relation is not transitive strong, jungle equilibria need not be Pareto optimal. Indeed, the allocations (g_1, g_2) and (g_2, g_1) are both jungle equilibria, but (g_1, g_2) is Pareto optimal whereas (g_2, g_1) is not.

When preferences are symmetric, however, *any* allocation is Pareto-optimal. That is because, when two agents exchange goods, at least one of them must be worse off. Under this assumption, therefore, there is no way exchanging goods can be Pareto-improving. Allocation by force, as was mentioned in the above quotation of Hobbes, is very likely.

Finally, we can show that where there is a transitive weak power relation there is a Pareto-optimal jungle equilibrium.[16]

4.3 Pareto-minimality

We have observed in section 3 that when preferences are the same across agents, a jungle equilibrium is Pareto-minimal. This is also the case when there is a transitive weak power relation. To wit, any reallocation of goods benefits at least one agent. We have also relied on example 1 to show that when preferences are not the same across agents a jungle equilibrium may not be a Pareto-minimal allocation. This conclusion also applies when there is a transitive weak power relation. Consider again example 1 where we now assume that $1 \sim 4$. One can check again that the allocation (g_2, g_4, g_1, g_3) is a jungle equilibrium. But as we have already observed, if we allocate good g_1 to player 2, and good g_4 to player 3, leaving the other agents with their equilibrium holdings, we decrease the payoffs of these two players. Thus the jungle equilibrium is not Pareto-minimal.

To study the Pareto-minimality of jungle equilibria, we shall first focus on the equilibrium studied in Proposition 4 because it would coincide with the unique equilibrium were the power relation be a strict order. The equilibrium in Proposition 4 is such that agent 1 (who is as powerful as any other agent) receives his best choice, then agent 2 (who is as powerful as any other agent except, perhaps, agent 1) receives his best choice in the set of remaining available goods, and so on. We shall say that this equilibrium is *associated* with the transitive weak power relation $\succsim$. The following Proposition provides a necessary and sufficient condition for this jungle equilibrium to be Pareto-minimal.

[16] This can be seen by using the allocation studied in the proof of Proposition 4. Since each agent receives the good that he ranks the highest in the set of goods left to him by the agents as powerful as him, there is no way one can devise a Pareto-improving allocation.

Corollary 1. *The jungle equilibrium associated with the transitive weak power relation* $\gtrsim$ *is Pareto-minimal if, and only if, for all integers* i *and* j *such that* $j < i$, $u_j\big(x(i)\big) > u_j\big(x(j)\big)$.

Proof. The arguments used to prove Proposition 1 apply directly. □

To address the Pareto-minimality of other kinds of jungle equilibria consider the next example.

Example 7 Assume the same setting as for example 4 except that now $1 \sim 3$. Suppose that an allocation $\boldsymbol{x}$ is a Pareto-minimal jungle equilibrium. Since it is a jungle equilibrium, we must have: $u_1\big(x(1)\big) > u_1\big(x(2)\big)$, and $u_2\big(x(2)\big) > u_2\big(x(3)\big)$. Furthermore, Pareto-minimality implies that: $u_2\big(x(1)\big) > u_2\big(x(2)\big)$, and $u_3\big(x(2)\big) > u_3\big(x(3)\big)$. Pareto-minimality also implies that *either*: $u_1\big(x(1)\big) > u_1\big(x(3)\big)$ and $u_3\big(x(1)\big) > u_3\big(x(3)\big)$, *or* $u_1\big(x(3)\big) > u_1\big(x(1)\big)$ and $u_3\big(x(3)\big) > u_3\big(x(1)\big)$. Notice that two given agents must rank two different goods in the same way. The first case (namely $u_1\big(x(1)\big) > u_1\big(x(3)\big)$ and $u_3\big(x(1)\big) > u_3\big(x(3)\big)$) is the jungle equilibrium taken into account in Corollary 1. The second case, however, suggests another necessary and sufficient condition for a jungle equilibrium to be Pareto-minimal.

Proposition 7. *Consider a jungle where there is transitive weak power relation. Then a jungle equilibrium is Pareto-minimal if, and only if, for two agents* i *and* j *either*

$$u_i\big(x(i)\big) > u_i\big(x(j)\big) \Rightarrow u_j\big(x(i)\big) > u_j\big(x(j)\big). \tag{3}$$

or

$$u_i\big(x(j)\big) > u_i\big(x(i)\big) \Rightarrow u_j\big(x(j)\big) > u_j\big(x(i)\big). \tag{4}$$

Proof. The proof essentially relies on an argument similar to the one used in the above example. □

Finally, the arguments put forward in section 2 to justify the notion of $\succ$-Pareto minimality also apply when the jungle has a transitive weak order: decreasing the payoff of an agent takes stronger people than him. Building on this remark we have

Proposition 8. *When the jungle has a transitive weak power relation $\gtrsim$ there is a $\succ$-Pareto-minimal jungle equilibrium.*

Proof. Consider the jungle equilibrium $\boldsymbol{x}$ associated with the transitive weak power relation $\gtrsim$. Using the proof of Proposition 2 one can show that there is actually no $\succ$-worsening of this allocation, and the result follows. $\square$

5 Conclusion

Exchanges are not always voluntary. Sometimes the more powerful just takes from the less powerful. When there is a transitive strong power relation, we have seen that jungle equilibria are not necessarily Pareto-minimal. That is, there is sometimes a way to decrease the welfare of an agent without increasing the welfare of another one. To put it another way, things could be worse. If one takes the power relation into account, however, jungle equilibria are $\succ$-Pareto-minimal. That is because agents can always resist Pareto-deteriorating reallocations of resources.

We have also observed that the power relation is not necessarily asymmetric. In this setting we have shown that jungle equilibria are not always unique nor Pareto-optimal. We have also shown, however, that when there is a transitive weak power relation (implying that two agents can be equally powerful) jungle equilibria exist, are Pareto-optimal and may be Pareto-minimal.

There are at least three natural ways to further the analysis of this paper. First, while we have restricted ourselves to the simplest case where there are as many agents as indivisible goods, one may consider the case where goods are multiple and divisible, as in the original paper by Piccione and Rubinstein. Second, other notions of power deserve consideration. For instance, the power relation could depend on the good which is in dispute (an agent may be stronger than another one regarding a certain class of goods, and weaker for another class). Third, it would be interesting to relate jungle economies to pillage games (Jordan, 2005, 2009), wherein the power relation is defined on coalitions of players.

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